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Aligned-fields magnetogasdynamic wakes

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The effect of compressibility is included here in a study of wakes created in two-dimensional, steady, aligned-fields, magnetogasdynamic flow past obstacles. The gas is assumed to be viscous, resistive, and thermally conducting. With the Oseen type of approximation as well as the magnetogasdynamic boundary-layer approximation, a great simplification in the formulation of wakes results. The boundary-layer equations, although linearized, still retain the coupling between the velocity, the magnetic, and the temperature fields. The solution of the magnetogasdynamic wake, in general, is a superposition of three individual-wake components, each satisfying a diffusion type of equation. Only one of them is capable of extending upstream. Hence the wake picture is generally characterized by a conventional downstream wake with also the possibility of the existence of an upstream one.

To be sure, the general features of the magnetogasdynamic wakes in aligned-fields flows are similar to those of an incompressible fluid; however, the flow now, instead of just being subalfvénic, must be subcritical for the upstream wake to occur. That is, the flow condition corresponding to the occurrence of the upstream wake is such that the sum of the square of the Mach number M_∞ and the square of the Alfvén number A_∞ is less than unity. Evidently it is the mechanism of magneto-sonic-wave propagation that modifies the transition of wakes from the Alfvénic point $A_\infty = 1$ (for an incompressible fluid) to the subcritical arc $A^2_\infty + M^2_\infty = 1$ in the Taniuti–Resler diagram.

1. Introduction

The remarkable behaviour of magneto-fluid-dynamic wakes has attracted the attention of many authors. Theoretical studies (Hasimoto 1959, 1960; Gourdine 1960; Glauert 1963; Clauser 1963) have revealed that in the steady, uniform, magnetohydrodynamic (incompressible) flow past obstacles of an unbounded fluid that is viscous and electrically conducting, there exist, in general, two wakes extending, respectively, in the directions of $U_\infty + \alpha_\infty$ and $U_\infty - \alpha_\infty$, where U_∞ is the free-stream velocity, and $\alpha_\infty = H_\infty/(4\pi\rho_\infty)^{\frac{1}{2}}$, the Alfvén velocity based on the unperturbed magnetic-field vector and density at infinity. In particular, if the flow is aligned-fields and subalfvénic; i.e. if U_∞ and α_∞ are parallel or anti-parallel, and $|\alpha_\infty| > |U_\infty|$, the surprising upstream wake occurs. Moreover, the remaining trailing wake disappears in the limit of either vanishing viscosity or resistivity. Lately a mechanism accounting the disappearance of a wake in this limit was proposed (Sakurai 1963). Results from the experimental side are equally exciting. An experiment on the magnetohydrodynamic flow past a Rankine body conducted in a mercury tow tank (Ahlstrom 1963) seems to have confirmed the incompressible, aligned-fields theory; an upstream wake appears provided the flow is subalfvénic.

Although magnetohydrodynamic wakes can mostly be considered as well known, their generalizations to include the effect of compressibility remains, to the author's knowledge, to be carried out. Using the Friedrichs pulse diagram, Sears (1960) pointed out that in the case of aligned-fields flow of an inviscid, compressible fluid of finite electrical conductivity, the transition of the downstream inviscid wake to the upstream, and vice versa, should take place at flow conditions corresponding to points on the subcritical arc $A_\infty^2 + M_\infty^2 = 1$ in the Taniuti–Resler diagram. The Alfvén number A_∞ is defined as the ratio of the free stream speed to the Alfvén speed at infinity; the Mach number M_∞ is the ratio of the free-stream speed to the sonic speed there. The same prediction was also mentioned by Resler & McCune (1960). To help decide whether these phenomena are real, we want to study them in a model having many of the properties of real conducting fluids, especially gases. Therefore, considerations of viscosity, thermal conductivity, resistivity, finite Prandtl number, and finite magnetic Prandtl number seem desirable. These were studied by Fan (1963).

Thus the intention of the present study is to investigate the magnetogasdynamic wakes created in the steady, uniform, two-dimensional, aligned-fields flow of an unbounded, ideal gas past impermeable bodies. The gas is treated as a single fluid and is assumed to be not only resistive but also viscous and thermally conducting. Furthermore, the gaseous particles are supposed to collide frequently enough everywhere in the flow so that the tensor character of the transport coefficients is insignificant. Turbulence is ignored. To analyse the set of linearized magnetogasdynamic equations without further simplification would be formidable, if not impossible. Even in limiting cases, e.g. aligned-fields flows with zero resistivity, the magnetogasdynamic fields fail to split into individual modes. However, for flows at large magnetic Reynolds number $R_m = 4\pi\sigma U_\infty L$, large Reynolds number $R_e = \rho_\infty U_\infty L/\mu$, and large Péclét number $P_c = C_p \rho_\infty U_\infty L/\kappa$, the diffusion due to resistivity, viscosity, and heat conductivity is important only in narrow zones within the flow field. Standing Alfvén waves are diffused and damped. The current-and-vortex sheet that surrounds the body in the case of aligned-fields flow is expanded into a thin magnetogasdynamic boundary layer of large vorticity and current. Hence a great simplification to the flow problem can be achieved through the use of the familiar boundary-layer approximation. At distances far from the body vorticity and current so generated propagate away in a thin wake or wakes and decline in intensity.

It should be pointed out that for aligned-fields magnetohydrodynamic flows theories leading to flow pictures different from the above-described have been proposed (e.g. Chester 1961; Ludford & Singh 1963). This apparent discrepancy remains unresolved (Stewartson 1963). The present investigation making use of the concept of aligned-fields boundary layer may shed some light on the problem.

2. Aligned-fields boundary-layer equations and formulation of wakes

The boundary-layer equations that will be presented in this section are applicable for steady, aligned-fields flows within boundary layers (including wakes) where the electromagnetic-body force is comparable with the viscous force. These categories of steady flows include (i) flows at large R_e with fluids of magnetic Prandtl number $Pr_m = O(1)$, where $Pr_m = R_m/R_e$, and (ii) flows at large R_e with infinite Pr_m. Clearly in the former case the viscid and the inviscid layers, being about the same thickness, are undistinguishable and, in fact, form a single layer of large vorticity and current, while in the latter case, in principle, the inviscid layer has shrunk to zero thickness and only the viscous one remains; however, the condition of the frozen-in magnetic field together with the non-slip condition at the body surface enables us to conclude again that electromagnetic and viscous forces are comparable. (The current sheet here is diffused by viscosity.) It should be noted that for real conducting fluids the fluid property Pr_m has very small value and Sears's boundary-layer equations are appropriate (Sears 1961). Although the limit of vanishing Pr_m represents very realistically the physical situation, it is of theoretical importance to understand how vorticity and current are convected in the far-flow field in fluids with Pr_m at other values of interest, namely infinite Pr_m and $Pr_m = O(1)$.

Assuming also that the conventional Prandtl number of the fluid Pr is $O(1)$, an order-of-magnitude analysis of the magnetogasdynamic equations leads immediately to the boundary-layer equations. The procedure is analogous to that used in obtaining the boundary-layer equations of non-conducting fluids except the magnetic-field-strength vector at the body surface is not known *a priori* in the case $Pr_m = O(1)$. As in the case of the inviscid boundary layer (Sears 1961), it is then further assumed that the magnitude of this magnetic field is at most of the order of $R_m^{-\frac{1}{2}}$ times the corresponding value given by the inviscid, ideal conducting solution. The body is assumed to be an insulator. Let $\bar{x}$ and $\bar{y}$ be the boundary-layer co-ordinates, i.e. co-ordinates measured along the body surface and perpendicular to it. The velocity $(\bar{u}, \bar{v})$ and the magnetic-field vector $(\bar{H}_x, \bar{H}_y)$ are decomposed in components along these directions. $\bar{\rho}$ denotes the density of the gas, $\bar{p}$ the pressure, $\bar{\tau}$ the temperature, μ the coefficient of viscosity, σ the electric conductivity, R the gas constant, C_p the constant pressure-specific heat, and κ the heat conductivity. The total pressure $\bar{P}$ is defined as the sum of the gas pressure and magnetic pressure, $\bar{p}+\bar{H}^2/8\pi$. $\bar{P}_1(\bar{x})$ represents the total pressure at the body surface given by inviscid, ideal conducting flow. The steady, two-dimensional boundary-layer equations in electromagnetic units, with the magnetic permeability of the gas equal to one, are then as follows (Pai 1962):

$$\partial \bar{\rho}\bar{u}/\partial \bar{x} + \partial \bar{\rho}\bar{v}/\partial \bar{y} = 0, \tag{1}$$

$$\partial \bar{H}_x/\partial \bar{x} + \partial \bar{H}_y/\partial \bar{y} = 0, \tag{2}$$

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{d\bar{P}_1(\bar{x})}{d\bar{x}} + \frac{\bar{H}_x}{4\pi}\frac{\partial \bar{H}_x}{\partial \bar{x}} + \frac{\bar{H}_y}{4\pi}\frac{\partial \bar{H}_x}{\partial \bar{y}} + \frac{\partial}{\partial \bar{y}}\left(\mu \frac{\partial \bar{u}}{\partial \bar{y}}\right), \tag{3}$$

$$\bar{P} = \bar{P}_1(\bar{x}), \tag{4}$$

$$\bar{u}\bar{H}_y - \bar{v}\bar{H}_x = -\frac{1}{4\pi\sigma}\frac{\partial \bar{H}_x}{\partial \bar{y}}, \tag{5}$$

$$\bar{\rho}\bar{u}C_p\frac{\partial\bar{\tau}}{\partial\bar{x}} + \bar{\rho}\bar{v}C_p\frac{\partial\bar{\tau}}{\partial\bar{y}} = \bar{u}\frac{\partial\bar{p}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{p}}{\partial\bar{y}} + \frac{\partial}{\partial\bar{y}}\left(\kappa\frac{\partial\bar{\tau}}{\partial\bar{y}}\right) + \mu\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2 + \frac{1}{16\pi^2\sigma}\left(\frac{\partial\bar{H}_x}{\partial\bar{y}}\right)^2, \tag{6}$$

$$\bar{p} = \bar{\rho}R\bar{\tau}. \tag{7}$$

Undoubtedly, the existence of irreversible transport mechanisms ensures that all disturbances originating in finite regions within the unbounded fluid are damped out completely at distances infinitely remote from these sources. Therefore there must exist a vast flow region extending inward from infinity, where the flow can be well described as a small perturbation of the uniform flow at infinity. Diffusion processes are significant there only in narrow zones called wakes, or diffuse waves. Vorticity and current confined in wakes are at least an order of magnitude greater than those outside. For fluids with $Pr_m = O(1)$, or $Pr_m = \infty$, the inertial, viscous, and electromagnetic-body forces are comparable and the boundary-layer approximation just mentioned prevails. It is sensible to postulate that in compressible flows the width of a wake also grows parabolically. Hence the variation of the total pressure along a wake can be overlooked in consistency to the boundary-layer approximation.

The non-dimensional version of equations (1) to (7) for wakes, linearized to first order in perturbed quantities, are

$$\partial\rho/\partial x + \partial u/\partial x + \partial v/\partial y = 0, \tag{8}$$

$$\partial H_x/\partial x + \partial H_y/\partial y = 0, \tag{9}$$

$$\frac{\partial u}{\partial x} - \frac{1}{A_\infty^2}\frac{\partial H_x}{\partial x} = \frac{\partial^2 u}{\partial y^2}, \tag{10}$$

$$p + \frac{\gamma M_\infty^2}{A_\infty^2}H_x = 0, \tag{11}$$

$$H_y - v = -\frac{1}{Pr_m}\frac{\partial H_x}{\partial y}, \tag{12}$$

$$\frac{\partial\tau}{\partial x} - \frac{\gamma-1}{\gamma}\frac{\partial p}{\partial x} = \frac{1}{Pr}\frac{\partial^2\tau}{\partial y^2}, \tag{13}$$

$$p - \rho - \tau = 0, \tag{14}$$

where γ is the ratio of the constant-pressure and the constant-volume specific heats of the gas. The dimensional quantities are non-dimensionalized respectively by their characteristic values; e.g. $x = \bar{x}/L$, $1+p = \bar{p}/p_\infty$. Furthermore, the parameter R_e has been absorbed in the normal boundary-layer co-ordinates, the normal velocity, and the normal magnetic-field components through magnifying them individually by a factor of $R_e^{\frac{1}{2}}$, e.g. $y = R_e^{\frac{1}{2}}\bar{y}/L$, $H_y = R_e^{\frac{1}{2}}\bar{H}_y/\bar{H}_\infty$. Viscous dissipation and Joule heating are second-order quantities and are neglected in equation (13).

The resulting differential equation in any single perturbed variable, say F, from equations (8) to (14) is

$$\left[A_\infty^2\left(\frac{\partial^2}{\partial y^2}-\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial y^2}-Pr\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial y^2}-Pr_m\frac{\partial}{\partial x}\right)-\gamma M_\infty^2 Pr_m\left(\frac{\partial^2}{\partial y^2}-\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial y^2}-\frac{Pr}{\gamma}\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\right.$$
$$\left.-Pr_m\left(\frac{\partial^2}{\partial y^2}-Pr\frac{\partial}{\partial x}\right)\frac{\partial^2}{\partial x^2}\right]F=0, \tag{15}$$

with the observation of the boundary condition that derivatives of all orders of the perturbation quantities vanish identically at distances infinitely far from the obstacle.

It is of great significance that in all physically possible situations, i.e. $Pr_m > 0$, $Pr > 0$, and $\gamma > 1$, the sixth-order differential operator can be factored into a product of three diffusion-type operators $W_{1+}W_{1-}W_2$ with real coefficients. Only one of the operators, W_{1-}, changes its character of diffusion as the sum $A_\infty^2+M_\infty^2$ alters across the value unity. In the limit of vanishing Mach number, (15) contains properly the incompressible result, (e.g. Gourdine 1960). If $Pr = 1$, the factorization is particularly simple and (15) can be written as

$$W_{1+}W_{1-}W_2F=0, \tag{16}$$

with explicitly

$$W_{1\pm}=\frac{\partial^2}{\partial y^2}-\tfrac{1}{2}A_\infty^{-2}[Pr_m(A_\infty^2+\gamma M_\infty^2)+A_\infty^2\pm\{[Pr_m(A_\infty^2+\gamma M_\infty^2)+A_\infty^2]^2$$
$$-4Pr_mA_\infty^2(A_\infty^2+M_\infty^2-1)\}^{\frac{1}{2}}]\frac{\partial}{\partial x},$$

and
$$W_2=\partial^2/\partial y^2-\partial/\partial x.$$

Clearly, it is the operator W_{1-} that causes negative diffusion when the flow becomes subcritical.

Similar differential equation for wakes in flows involving gases of infinite Pr_m (viscous but perfectly conducting gases) can be obtained by formally taking this limit in (15). The result is

$$W_{1+}W_{1-}F=0, \tag{17}$$

where the operators $W_{1\pm}$ denote

$$\frac{\partial^2}{\partial y^2}-\tfrac{1}{2}(A_\infty^2+\gamma M_\infty^2)^{-1}[A_\infty^2+\gamma M_\infty^2+Pr(A_\infty^2+M_\infty^2)-1\pm\{[A_\infty^2+\gamma M_\infty^2$$
$$+Pr(A_\infty^2+M_\infty^2)-1]^2-4Pr(A_\infty^2+\gamma M_\infty^2)(A_\infty^2+M_\infty^2-1)\}^{\frac{1}{2}}]\frac{\partial}{\partial x}.$$

Again the coefficients in $W_{1\pm}$ are real for all physically possible circumstances and only W_{1-} is capable of changing its character of diffusion across the flow conditions $A_\infty^2+M_\infty^2 = 1$. Furthermore, if $Pr = 1$, (17) becomes

$$\left(\frac{\partial^2}{\partial y^2}-\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial y^2}-\frac{A_\infty^2+M_\infty^2-1}{A_\infty^2+\gamma M_\infty^2}\frac{\partial}{\partial x}\right)F=0. \tag{18}$$

Of laboratory interest is the limiting case when resistivity alone is assumed to play the role of diffusion—the so-called inviscid wake. Neglecting the viscous term in equation (10) and replacing (13) and (14) by the isentropic relation $p = \gamma\rho$, the resulted inviscid-wake equation is simply

$$W_{1-}F = 0, \tag{19}$$

where

$$W_{1-} = \frac{\partial^2}{\partial y^2} - \frac{A_\infty^2 + M_\infty^2 - 1}{A_\infty^2}\frac{\partial}{\partial x}.$$

Analogously, in equation (19) the parameter R_m has been made implicit in the stretched y co-ordinate, etc. If the gas is allowed to exhibit thermal diffusion comparable to resistive diffusion while retaining its inviscid nature, it can be shown that the resulted differential operator is now a product of two diffusion operators with real coefficients, and again only one of them appears with negative diffusion at subcritical flows.

Nothing definite can yet be concluded at this stage, since (15), (17), and (19) generally yield necessary conditions that solutions for wakes must satisfy. Nevertheless, they all indicate the existence of a mechanism of negative diffusion at subcritical-flow régimes, and therefore strongly suggest the possibility of an upstream wake in compressible, aligned-fields flows. In fact, it will be shown later that solutions of this type do exist. It is of some interest to note that the orders of these differential equations are directly proportional to the number of diffusive mechanisms exhibited by the assumed gas.

3. Similarity solutions for wakes

To demonstrate theoretically the existence of an upstream magnetogasdynamic wake, a class of similarity solutions will be obtained for equations (8) to (14). As will be seen later, the similarity solutions represent wakes in aligned-fields flows past a thin, symmetrical, cylindrical body at zero incidence. The body must also be symmetrical in thermal and magnetic boundary conditions. Evidently, effects of the distant body on the wake profiles should be characterized at least to some extent by the gross aerodynamic force (wave drag excluded) experienced by the body and the gross energy exchange between the body and the fluid. That such characterization is sufficient, namely that the gross quantities determine completely the wake profiles, can be easily seen in conventional fluid dynamics, as the temperature and velocity fields are uncoupled in the linearized boundary-layer approximation even in compressible flows. Contrarily, the resistive diffusion in magneto-fluid-dynamics, if present, makes the characterization incomplete either by introducing additional wake components or for lack of a mechanism analogous to viscosity in producing the drag. Therefore in the attempt to discuss the magnetogasdynamic wakes without knowing in detail the near flow field, a degree of arbitrariness in the final wake profiles is unavoidable for certain cases. Disregarding this, the similarity solutions do serve the purpose of showing the appearance of an upstream wake at subcritical flows.

The similarity variable $y/x^{\frac{1}{2}}$ for viscous wakes in conventional fluid dynamics finds its application also in aligned-fields magnetogasdynamics—the slight modification being that the absolute value of the streamwise co-ordinate $|x|^{\frac{1}{2}}$

must now be taken in the denominator. The origin of the co-ordinate system is assumed to be located somewhere within the body. Let

$$(u, \tau, H_x) = |x|^{-\frac{1}{2}}(f, g, h), \tag{20}$$

where f, g, and h are functions of the parameter $\eta = y/|x|^{\frac{1}{2}}$ only. It can easily be seen that the mathematical problem now reduces to the determination of f, g, and h which satisfy simultaneously the following three ordinary differential equations with appropriate boundary conditions and constraints:

$$-\operatorname{sgn} x\eta(f - A_\infty^{-2}h) = 2\{f' - f'(0)_\pm\}, \tag{21}$$

$$\operatorname{sgn} x\eta\left(f - g - \frac{A_\infty^2 + \gamma M_\infty^2}{A_\infty^2}h\right) = \frac{2}{Pr_m}h', \tag{22}$$

$$-\operatorname{sgn} x\eta\left\{g + \frac{(\gamma-1)M_\infty^2}{A_\infty^2}h\right\} = \frac{2}{Pr}\{g' - g'(0)_\pm\}, \tag{23}$$

where the prime denotes differentiation with respect to η. Upper and lower signs in subscripts are appropriate, respectively, for $\operatorname{sgn} x = d|x|/dx = 1$ and $\operatorname{sgn} x = -1$. Other perturbed quantities, namely p, ρ, v, and H_y can be expressed in terms of f, g, and h.

To satisfy the Ohm's law (equation (22)), $h'(0)_\pm$ must vanish for all finite values of Pr_m. If the 'steady' state is reached, it is necessary that solutions for wakes be compatible with three constraints: (I) The surface integral of the momentum-flux tensor over the surface to a control volume enclosing the obstacle, which is everywhere distant from the obstacle, should be independent of the volume chosen and numerically equal to the aerodynamic force exerted by the fluid on the body but opposite in sign. (II) Similarly, the surface integral of the energy-flux-density vector is invariant with respect to the volume chosen and equal to the net energy transfer from the body to the fluid. (III) Furthermore, an analogous condition must exist in relating the moment acting on the body to the surface integral of the moment of the momentum-flux tensor. Since the flow outside the wakes is considered to be ideal, constraints (I) and (II) lead to

$$Re^{\frac{1}{2}}C_{DW} = -\left[\int_{-\infty}^{\infty}(f - A_\infty^{-2}h)\,d\eta\right]_+ + \left[\int_{-\infty}^{\infty}(f - A_\infty^{-2}h)\,d\eta\right]_-, \tag{24}$$

$$Re\,C_{LW} = -\frac{A_\infty^2 - 1}{2A_\infty^2}\left[\left(|x|^{-\frac{1}{2}}\int_{-\infty}^{\infty}\eta h\,d\eta\right)_+ + \left(|x|^{-\frac{1}{2}}\int_{-\infty}^{\infty}\eta h\,d\eta\right)_-\right] \approx 0, \tag{25}$$

$$Re^{\frac{1}{2}}C_E = -\left\{\int_{-\infty}^{\infty}[f + (\gamma-1)^{-1}M_\infty^{-2}g]\,d\eta\right\}_+ + \left\{\int_{-\infty}^{\infty}[f + (\gamma-1)^{-1}M_\infty^{-2}g]\,d\eta\right\}_-, \tag{26}$$

where only leading terms have been kept and the limits of integration have been extended from the edges of the wakes to infinity. To justify the extension, the class of functions that are admissible to the solutions for f, g, and h must not only assure the existence of all the integrals but also die out rapidly enough at large $|\eta|$. C_{DW} and C_{LW} are, respectively, the drag and the lift coefficients contributed by wakes, while the non-dimensional coefficient C_E represents the net energy transfer between the fluid and the body. The subscript '$_+$' denotes integration over a section across the downstream wake at $x \gg 1$; '$_-$' the upstream wake at $x \ll -1$. From (24) and (26) it is clear that u, H_x, and τ in their similarity forms yield values of C_{DW} and C_E independent of the choice of surface of integration. The total lift is presumably related to the circulation and the magnetic circulation around the body. In view of the postulate of parabolic-wake growth, C_{LW} must be negligible in comparison with the total lift coefficient. At first glance this can be achieved either by limiting the applicability of the similarity solutions at large enough $|x|$ or by requiring the integrals in (25) to vanish identically as in the cases of symmetrical flows where the total lift coefficient itself vanishes. The vanishing of the integrals is in agreement with the 'symmetry' condition that demands f, g, and h to be even in η, provided the axis of symmetry is now taken to be the x axis. Furthermore, wake profiles with $f'(0)_\pm = g'(0)_\pm = h'(0)_\pm = 0$ are compatible with Ohm's law, the momentum equation, and the energy equation. The constraint (III), however, restricts the similarity solution to flows that are symmetrical, as integrals of the type $|x|^{\frac{1}{2}}\int_{-\infty}^{\infty}\eta f d\eta$, for example, appear in the expression for the moment coefficient. It can easily be verified that all these integrals vanish again if f, g, and h are even. Finally, the boundary condition at infinity (in the physical plane) states that all perturbation quantities and their derivatives must vanish there.

(i) *Wakes without resistive diffusion*

We shall consider first solutions for large R_e and infinite Pr_m with (1) $Pr = 1$, and (2) arbitrary Pr, i.e. $Pr = O(1)$.

With $f'(0)_\pm = g'(0)_\pm = 0$ a straightforward manipulation of equations (21), (22), and (23) gives the following solution for the case Pr equal to one:

$$h = \frac{A_\infty^2}{A_\infty^2+\gamma M_\infty^2} A_1 \exp\left[-\frac{A_\infty^2+M_\infty^2-1}{4(A_\infty^2+\gamma M_\infty^2)}\operatorname{sgn} x\eta^2\right], \tag{27}$$

$$f = A_2 \exp(-\tfrac{1}{4}\eta^2) + \frac{A_1}{(\gamma-1)M_\infty^2+1}\exp\left[-\frac{A_\infty^2+M_\infty^2-1}{4(A_\infty^2+\gamma M_\infty^2)}\operatorname{sgn} x\eta^2\right], \tag{28}$$

$$g = A_2 \exp(-\tfrac{1}{4}\eta^2) - \frac{(\gamma-1)M_\infty^2}{(\gamma-1)M_\infty^2+1} A_1 \exp\left[-\frac{A_\infty^2+M_\infty^2-1}{4(A_\infty^2+\gamma M_\infty^2)}\operatorname{sgn} x\eta^2\right], \tag{29}$$

A_1 and A_2 are constants of integration. The factor sgn x that appears in equations (27), (28), and (29) must take the value

$$+1 \quad \text{if} \quad A_\infty^2+M_\infty^2 > 1 \quad \text{and} \quad -1 \quad \text{if} \quad A_\infty^2+M_\infty^2 < 1.$$

That is, wake components containing sgn x in their exponents extend upstream in subcritical-flow régimes. f, g, and h are clearly even functions of η. The constants A_1 and A_2 are related uniquely to C_{DW} and C_E through (24) and (26). The relations are

$$A_1 = \frac{1}{2}\left(\frac{A_\infty^2+\gamma M_\infty^2}{|A_\infty^2+M_\infty^2-1|}\frac{R_e}{\pi}\right)^{\frac{1}{2}}[(\gamma-1)M_\infty^2(C_E-C_{DW})-C_{DW}], \tag{30}$$

$$A_2 = -\frac{(\gamma-1)M_\infty^2}{2[(\gamma-1)M_\infty^2+1]}\left(\frac{Re}{\pi}\right)^{\frac{1}{2}} C_E. \tag{31}$$

The rate of energy exchange between the gas and the body as will be noted below has intricate effects on the structure of aligned-fields magnetogasdynamic wakes. Let $\bar{C}_E$ denote $\{1+(\gamma-1)^{-1}M_\infty^{-2}]C_{DW}$. If the body acts as an energy sink with a rate of absorption $C_E=\bar{C}_E$, current diffusion occurs nowhere in the far flow field. In other words, there is no perturbation in the magnetic field. The wake structure is conventional, as if the magnetogasdynamic interaction were totally absent. Moreover, in the limit $C_E=\bar{C}_E$ the upstream wake disappears at subcritical flows. Now, if $C_E>\bar{C}_E$ there is a magnetic-field increment in contrast to the deficiency that occurs when $C_E<\bar{C}_E$; hence current, if any, at any point in the wake region flows in opposite directions depending on $C_E \lesseqgtr \bar{C}_E$. The resolution of the solutions into up- and downstream portions at subcritical flows enables each wake component to be examined individually. The following remarks are restricted to subcritical flows: (1) Vorticity is shed away both up- and downstream, whereas current generated due to the presence of the body can only diffuse upstream. The current-free trailing wake vanished in the limit $C_E=0$. (2) The downstream wake enjoys both velocity and temperature increments when the gas gains energy steadily from the body, i.e. $C_E<0$. Therefore the density of the gas in the trailing wake must suffer, in proportion, a deficiency so that the gas pressure is constant and the condition of the frozen-in magnetic field is fulfilled. The reverse ($C_E>0$) is also true. (3) As for the upstream wake, the velocity shows an increment or a deficiency corresponding to $C_E>\bar{C}_E$ or $C_E<\bar{C}_E$. The temperature there behaves oppositely. (4) As a direct consequence of positive drag, the present analysis denies the existence of wakes with velocity increments both up- and downstream or with temperature decrements upstream and increments downstream.

For arbitrary Prandtl number, it can be verified that the solution takes the form

$$f=A_3\exp(-K_+\eta^2)+A_4\exp(-K_-\operatorname{sgn}x\eta^2), \tag{32}$$

$$g=[(A_\infty^2+\gamma M_\infty^2)(4K_+-1)+1]A_3\exp(-K_+\eta^2) + [(A_\infty^2+\gamma M_\infty^2)(4K_--1)+1]A_4\exp(-K_-\operatorname{sgn}x\eta^2), \tag{33}$$

$$h=A_\infty^2[(1-4K_+)A_3\exp(-K_+\eta^2)+(1-4K_-)A_4\exp(-K_-\operatorname{sgn}x\eta^2)], \tag{34}$$

where $4K_\pm$ are, respectively (except for sign) the coefficients of the x-derivatives in the operators $W_{1\pm}$ in equation (17). The K_+-wake component always travels downstream, while the K_--wake component propagates upstream when

$$A_\infty^2+M_\infty^2<1.$$

The constants of integration A_3 and A_4 are related to C_{DW} and C_E as follows:

$$\begin{aligned}(A_\infty^2+M_\infty^2-1)&[(A_\infty^2+\gamma M_\infty^2)(8K_+-1)-Pr(A_\infty^2+M_\infty^2)+1]A_3\\ &=(A_\infty^2+\gamma M_\infty^2)[4(A_\infty^2+\gamma M_\infty^2)K_--(A_\infty^2+M_\infty^2-1)][K_+(Re/\pi)]^{\frac{1}{2}}C_{DW}\\ &\quad\mp(\gamma-1)M_\infty^2[Pr(A_\infty^2+\gamma M_\infty^2)(A_\infty^2+M_\infty^2-1)K_-(Re/\pi)]^{\frac{1}{2}}C_E,\end{aligned} \tag{35}$$

$$\begin{aligned}\mp(A_\infty^2+M_\infty^2-1)&[(A_\infty^2+\gamma M_\infty^2)(8K_+-1)-Pr(A_\infty^2+M_\infty^2)+1]A_4\\ &=(A_\infty^2+\gamma M_\infty^2)[4(A_\infty^2+\gamma M_\infty^2)K_+-(A_\infty^2+M_\infty^2-1)][|K_-|(Re/\pi)]^{\frac{1}{2}}C_{DW}\\ &\quad-(\gamma-1)M_\infty^2[Pr(A_\infty^2+\gamma M_\infty^2)|A_\infty^2+M_\infty^2-1|K_+(Re/\pi)]^{\frac{1}{2}}C_E,\end{aligned} \tag{36}$$

where upper signs are appropriate for $A_\infty^2 + M_\infty^2 > 1$ and lower signs for $A_\infty^2 + M_\infty^2 < 1$. If $Pr = 1$, equations (32) to (36) check properly with the previous solution for $Pr = 1$.

For gases with $Pr \neq 1$, it should be noted that current-free flow in wake regions is no longer possible. Associated with each temperature- (or velocity-) wake component, there exists also a magnetic one, in general. A_3 (or A_4) vanishes if C_E is such as to cancel the contribution of C_{DW} in the right-hand side of equation (35) (or equation (36)). Hence, by properly adjusting the rate of net energy exchange between the body and the gas, subcritical flows with only an upstream wake or with only a downstream wake can be realized as special cases in gases with infinite conductivity. Further effects of C_E on the structure of wakes can be discussed along the same line as before.

The unique determination of the constants of integration in terms of the aerodynamic coefficients C_{DW} and C_E implies that the asymptotic behaviour of the flow fields in wakes is independent of the details of the near-flow field, e.g. the shape of the body. It will become clear later that this statement is invalid in cases when R_m is finite.

(ii) *Wakes with resistive diffusion*

We shall consider under this subtitle aligned-fields magnetogasdynamic wakes in gases (1) with resistive, viscous, and thermal diffusions mutually comparable, i.e. Pr_m and Pr both of the order of unity; and (2) with resistive diffusion only, namely $Pr_m = 0$.

If Pr_m is of the order of one and Pr is equal to one, the solution for wakes can easily be obtained:

$$h = A_\infty^2[(1-4L_+)A_5 \exp(-L_+\eta^2) + (1-4L_-)A_6 \exp(-L_- \operatorname{sgn} x\eta^2)], \tag{37}$$

$$f = A_7 \exp(-\tfrac{1}{4}\eta^2) + A_5 \exp(-L_+\eta^2) + A_6 \exp(-L_- \operatorname{sgn} x\eta^2), \tag{38}$$

$$g = A_7 \exp(-\tfrac{1}{4}\eta^2) - (\gamma-1)M_\infty^2[A_5 \exp(-L_+\eta^2) + A_6 \exp(-L_- \operatorname{sgn} x\eta^2)]. \tag{39}$$

The expressions $4L_\pm$ are, respectively (except for sign) the coefficients of the x-derivatives in the operators $W_{1\pm}$ in (16). A_5, A_6, and A_7 are constants of integration.

Clearly, the similarity solutions (37), (38), and (39), demonstrate the possibility of the existence of an upstream wake—the L_--wake component, in subcritical flows. Nevertheless, the two constraints, (24) and (26), now fail to determine uniquely the constants A_5, A_6, and A_7. Hence the complete solution in the present case has to be determined by matching the near-field solution at one point. That is, in addition to the gross quantities C_{DW} and C_E, the wakes bear invariantly the influence of the details of the near-flow field regardless of how far away from the body the similarity solutions are applied. It is interesting to note that A_7 relates to C_E alone

$$A_7 = -\frac{(\gamma-1)M_\infty^2}{2[(\gamma-1)M_\infty^2+1]}\left(\frac{Re}{\pi}\right)^{\frac{1}{2}} C_E. \tag{40}$$

For arbitrary Pr, there is also a third-wake component for the magnetic field which always travels downstream. Only one among the three wake components gives rise to the solution that extends upstream when $A_\infty^2 + M_\infty^2 < 1$.

If resistive diffusion alone is present, the wake is inviscid in nature. R_m now plays the role of R_e in the formulation and is absorbed in the stretched y coordinate. Hence for the inviscid wake the Ohm's law in its similarity form is

$$\operatorname{sgn} x\eta \left(f - g - \frac{A_\infty^2 + \gamma M_\infty^2}{A_\infty^2} h \right) = 2h'. \tag{41}$$

With the right-hand sides of (21) and (23) replaced by zero, the solution for the inviscid wake is readily obtainable:

$$f = A_8 \exp\left(-\frac{A_\infty^2 + M_\infty^2 - 1}{4A_\infty^2} \operatorname{sgn} x\eta^2 \right), \tag{42}$$

$$g = -(\gamma - 1) M_\infty^2 A_8 \exp\left(-\frac{A_\infty^2 + M_\infty^2 - 1}{4A_\infty^2} \operatorname{sgn} x\eta^2 \right), \tag{43}$$

$$h = A_\infty^2 A_8 \exp\left(-\frac{A_\infty^2 + M_\infty^2 - 1}{4A_\infty^2} \operatorname{sgn} x\eta^2 \right), \tag{44}$$

A_8 is the constant of integration.

As a whole, the downstream inviscid wake flips over to the upstream direction (with a possible variation in the amplitude A_8) when the flow changes from supercritical to subcritical. The inviscid wake contributes nothing to the lift, to the moment, nor to the energy transfer, since the flow is symmetrical and thermally non-conducting. That C_{DW} also vanishes would seem astonishing at the first sight. However, this implies merely that C_{DW} must be no greater than $O(R_m^{-1})$. This conclusion is consistent with an estimate based on the inviscid boundary-layer theory, assuming that this theory is uniformly valid over the entire body surface.

In this connexion it should be mentioned that an estimate made by Lary (1962) gives $C_{DW} = O(R_m^{-\frac{1}{2}})$. This was based on the observation that the rate of energy dissipated in joule heating is of this order. The apparent conflict between our estimate and his has not yet been explained. It may imply that the boundary-layer theory cannot be valid over the whole body, and that a pressure drag of $O(R_m^{-\frac{1}{2}})$ necessarily occurs. Furthermore there may appear in the flow field a non-diffusive 'vortex trail' which extends indefinitely downstream, similar to what Tamada (1962) has described. If a vortex trail exists, the linearized treatment will not hold there. In any case the detailed description of the wake for a particular configuration calls for matching to the near-flow field solution, and the present treatment of the wake is valid for regions where the perturbations are small.

4. Conclusions and remarks

The possibility of an upstream wake in aligned-fields magnetogasdynamic flow past obstacles has been considered. The upstream wake occurs invariably in subcritical-flow régimes for ideal gases with arbitrary Pr, and with Pr_m infinite, of the order of one, or equal to zero. In the incompressible limit the transition of the downstream wake to upstream and vice versa is again predicted to take place at the Alfvén point, in full agreement with previous work on incompressible flows. The similarity solutions indicate explicitly the parabolic growth of aligned-fields magnetogasdynamic wakes, and hence justify *a posteriori* the basic postulate that the variation of the total pressure along wakes is insignificant. The procedure employed here in obtaining solutions is equivalent to the method of superposing the elementary solutions of individual diffusion operator in equations (15), (17), or (19). In general, the number of wake components presented in the solution is equal to the number of diffusive properties that the gas is assumed to possess, and only one wake component is capable of travelling upstream when the flow becomes subcritical.

A striking effect of compressibility is that C_E, the rate of net energy transfer between the body and the gas, has a profound influence on the structure of aligned-fields magnetogasdynamic wakes. By adjusting C_E properly, the upstream wake can be made to vanish in subcritical flows involving gases of infinite Pr_m. Consequently, the necessary and sufficient correspondence between the presence of the forward, degenerate wave (as can be seen from the Friedrichs pulse diagram) and the appearance of the upstream wake (Sears 1960) is strictly valid for the inviscid aligned-field magnetogasdynamic wake.

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